Last time: ①

$K/\mathbb{Q}$ finite, $C \in Cl_K$  $t \geq 1$

$N_C(t) := \#\{0 \neq I \subseteq \mathcal{O}_K \mid I \in C, N(I) \leq t\}$,

$= \#\{x \in J \mid N_{K/\mathbb{Q}}(x) \leq t \cdot N(J)\} / U_K$

$J^{-1} \in C$ fract.

Then $N_C(t) = \dfrac{2^{r_1}(2\pi)^{r_2} \cdot R_K}{w \cdot \sqrt{|\Delta_K|}} \cdot t + O(t^{1-\frac{1}{n}}$

$R_K = \left|\det\left(\frac{1}{r_1+r_2}(1,\dots,1), \ell(u_1), \dots, \ell(u_s)\right)\right|$

$u_1, \dots, u_s \in U_K$, basis modulo $W_K$

$s := r_1 + r_2 - 1$

$$(\sim R_K = \frac{1}{\sqrt{r_1+r_2}} \cdot \mathrm{vol}(H/\ell(U_K)), \quad H \subseteq \mathbb{R}^s,$$

$$H = \{x \mid \sum x_i = 0\}$$

$$w = \# W_K$$

If $K/\mathbb{Q}$ imag. quadr., $\iota : K \hookrightarrow \mathbb{C}$

$$\Rightarrow N_c(t) = \frac{1}{w} \#(\iota(\mathfrak{f}) \cap B_{\sqrt{tN(\mathfrak{f})}})$$

$B_{\sqrt{tN(\mathfrak{f})}}$: disc of radius $\sqrt{tN(\mathfrak{f})}$

$$\approx \frac{\frac{1}{w}\mu(B_{\sqrt{tN(\mathfrak{f})}})}{\mathrm{vol}(\mathbb{C}/\iota(\mathfrak{f}))}$$

$\approx$: $O(t^{\frac{1}{2}})$

$$= \frac{2\pi \cdot t}{w \cdot \sqrt{|\Delta_K|}}$$

Assume now $K/\mathbb{Q}$ real quadratic

$$\iota : K \to \mathbb{R}^2$$

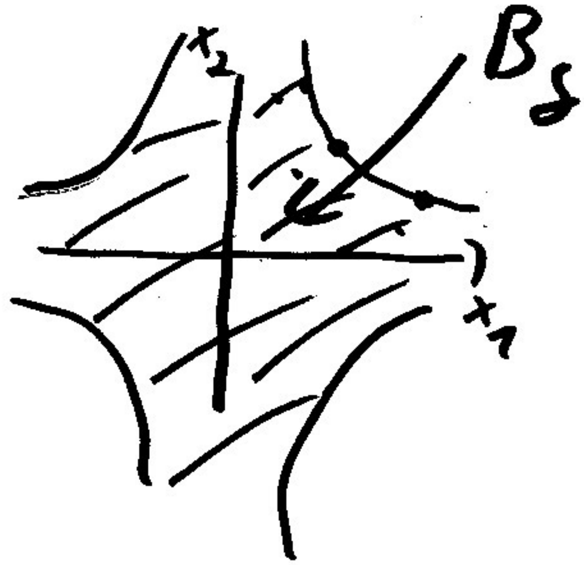
$$x \mapsto (\sigma_1(x), \sigma_2(x))$$

$|N_{K/\mathbb{Q}}(x)| = |m(\iota(x))|$

where $m(x_1, x_2) = x_1 \cdot x_2$

For $\delta \geq 0$ set

$$B_\delta := \{ (x_1, x_2) \in \mathbb{R}^2 \mid |m(x_1, x_2)| \leq \delta \}$$




Problem:

$\iota(\mathfrak{J}) \cap B_{t N(\mathfrak{J})}$

infinite, $U_K := \mathcal{O}_K^\times$

acts freely

Recall: $\ell : U_K \xhookrightarrow{\iota} (\mathbb{R}^\times)^2 \xrightarrow{\log} \mathbb{R}^2$

$$(x_1, x_2) \mapsto (\log|x_1|, \log|x_2|)$$

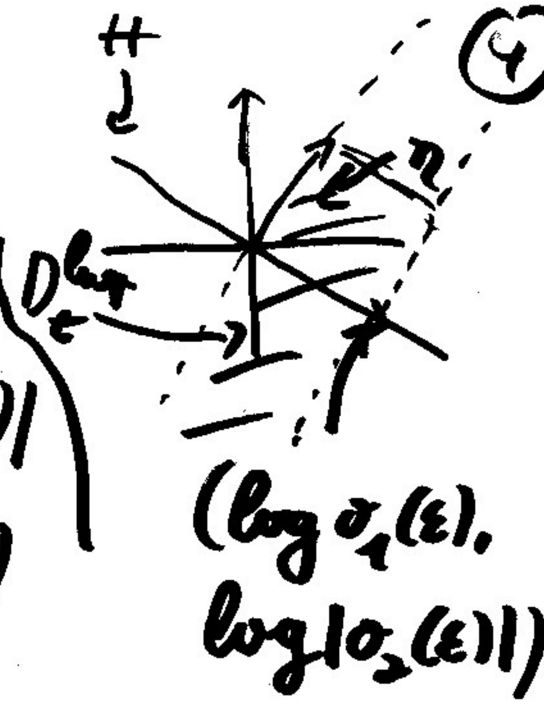
$\ell(U_K) \subseteq H := \{ (y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 = 0 \}$

full lattice

Fix $\varepsilon \in U_K$, s.t. $U_K / W_K = \langle \varepsilon \rangle$

wlog $\sigma_1(\varepsilon) > 1$

Set $\eta := \frac{1}{2}(1,1)$




& $D_t^{\log} = \{ t_0 \eta + t_1 (\log \sigma_1(\varepsilon), -\log \sigma_1(\varepsilon)) \mid$

$t_0 \in (-\infty, \log t \cdot N(J))$

$t_1 \in [0,1) \}$

$\Rightarrow \quad D_t := \mathrm{Log}^{-1}(D_t^{\log}) \subseteq B_{tN(J)}$

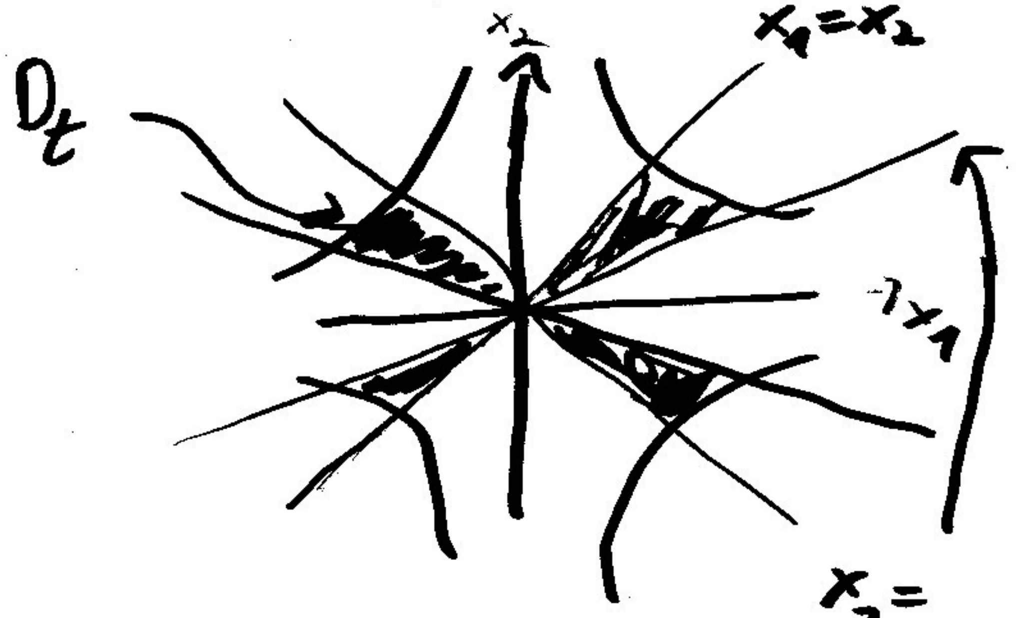
fund. domain for the action of $(\sigma_1(\varepsilon), \sigma_2(\varepsilon))$ on $B_{tN(J)}$

Moreover, $D_t = \{ (x_1, x_2) \in (\mathbb{R}^*)^2 \mid$

$\log|x_1| + \log|x_2| \leq \log t \cdot N(J),$

$\log|\sigma_1(\varepsilon)| \cdot \log|x_1| - \log|\sigma_1(\varepsilon)| \cdot \log|x_2| \in [0, 2 \cdot \log|\sigma_1(\varepsilon)|^2) \}$

$$= \{ (x_1, x_2) \in (\mathbb{R}^*)^2 \mid |x_1| \cdot |x_2| \leq t \cdot N(\mathfrak{f}), \; 1 \leq \frac{|x_1|}{|x_2|} < \sigma_1(\varepsilon)^2 \}$$




$$\Rightarrow \# \{ (\mathfrak{r}(\mathfrak{f}) \cap B_{t N(\mathfrak{f})}) / U_K \}$$

$$= \# \{ (\mathfrak{r}(\mathfrak{f}) \cap D_t) / W_K \}$$

$$\approx \frac{\mu(D_t)}{w \cdot \mathrm{vol}(\mathbb{R}^2 / \mathfrak{r}(\mathfrak{f}))}$$

up to $O(t^{\frac{1}{2}})$

Now, $$\mu(D_t) = 4 \int\limits_{\substack{0 < y_1, y_2 \\ y_1 \cdot y_2 \leq t \cdot N(\gamma) \\ 1 \leq \frac{y_1}{y_2} < \sigma_1(\varepsilon)^2}} dy_1\, dy_2$$

$$\underset{y_i = e^{x_i}}{=} 4 \int\limits_{\substack{x_1 + x_2 \leq \log t \cdot N(\gamma) \\ 0 \leq x_1 - x_2 < 2 \cdot \log \sigma_1(\varepsilon)}} e^{x_1 + x_2}\, dx_1\, dx_2$$

$$\underset{\substack{x_1 + x_2 = u \\ x_1 - x_2 = v}}{=} 4 \int\limits_{\substack{u \leq \log t \cdot N(\gamma) \\ 0 \leq v < 2 \log \sigma_1(\varepsilon)}} e^{u} \cdot \frac{1}{2}\, du\, dv$$

$$\left( \Leftrightarrow x_1 = \frac{u+v}{2}, \quad x_2 = \frac{u-v}{2} \right)$$

$$= 2 \cdot t \cdot N(\gamma) \cdot 2 \cdot \log \sigma_1(\varepsilon)$$

Altogether,

$$\frac{\mu(D_t)}{w \cdot \mathrm{vol}(\mathbb{R}^2/\pi(\mathfrak{J}))} = \frac{4 \cdot t \cdot N(\mathfrak{J}) \cdot \log \sigma_1(\varepsilon)}{2 \cdot \sqrt{|\Delta_K|} \cdot N(\mathfrak{J})}$$

$$= \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot R_K}{w \cdot \sqrt{|\Delta_K|}} \cdot t$$

(bec. $R_K = \left| \det \begin{pmatrix} \frac{1}{2} & \log \sigma_1(\varepsilon) \\ \frac{1}{2} & -\log \sigma_1(\varepsilon) \end{pmatrix} \right|$

$= \log \sigma_1(\varepsilon)$

General case: $K/\mathbb{Q}$ finite

$0 \neq \mathfrak{J} \subseteq K$ fract. ideal, $C = [\mathfrak{J}^{-1}]$

$K \overset{\pi}{\hookrightarrow} \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$,

$$\mathrm{Log}: (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \longrightarrow \mathbb{R}^{r_1 + r_2}$$

$$(y, z) \longmapsto (\log|y_i|, 2\log|z_i|)$$

Set $X_t := \{ (y,z) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \mid$

$$N(y,z) := \prod_{i=1}^{r_1} |y_i| \cdot \prod_{j=1}^{r_2} |z_j|^2 \le t \cdot N(\mathfrak{J}) \}$$

$X_t^* := X_t \cap (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$

$(\Rightarrow \; \mathfrak{J} \cap \pi^{-1}(X_t^*) = \{ x \in \mathfrak{J} \setminus \{0\} \mid$
$|N_{K/\mathbb{Q}}(x)| \le t \cdot N(\mathfrak{J}) \})$

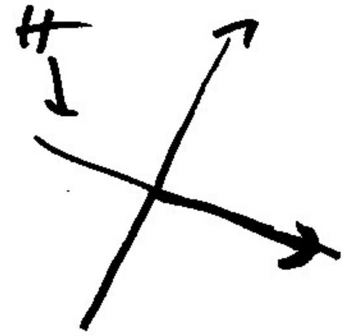
Need to find fund. domain for action of $U_K$ on $X_t^*$, more prec.

let $u_1, \dots, u_s \in U_K$, s.t. $u_1, \dots, u_s$ are a basis of $U_K / W_K$

$s := r_1 + r_2 - 1$

Set $F := \{ t_1 \cdot \ell(u_1) + \dots + t_s \ell(u_s) \mid t_1, \dots, t_s \in [0,1) \}$

fund. domain for action of $\ell(V_K)$ on $H \subseteq \mathbb{R}^{r_1+r_2}$ dfd by $\sum x_i = 0$




Set $N := \{ (x,z) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \mid N(x,z) = 1 \}$

Let $v : (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2} \to N$

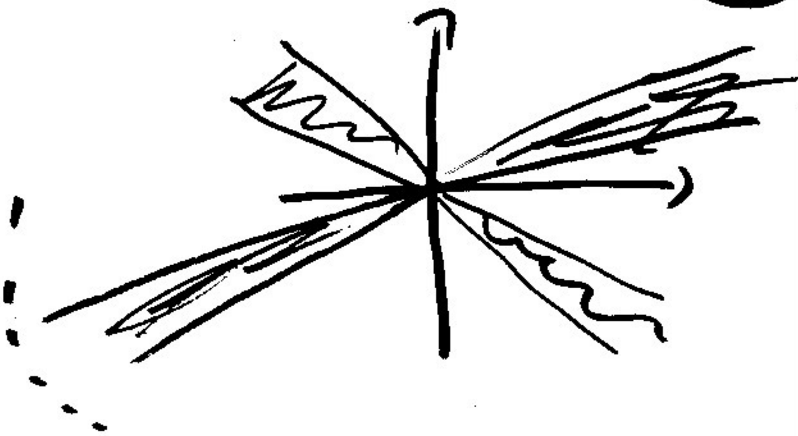
$$(x,z) \longmapsto N(x,z)^{-\frac{1}{n}} \cdot (x,z)$$

$\Rightarrow$ $D := v^{-1}(\mathrm{Log}_N^{-1}(F))$, $\mathrm{Log}_N : N \to H$

is a fund domain for $V_K \curvearrowright (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2}$

Set $D_t := D \cap X_t^*$

$= \{(x,z) \in D \mid N(x,z) \le t \cdot N(y)\}$



Upshot:

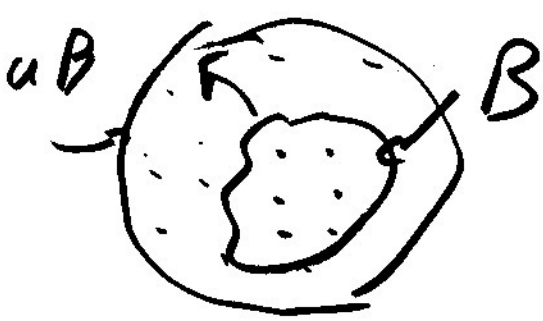
$$N_c(t) = \frac{\#(\pi(y) \cap D_t)}{w} \underset{O(t^{1-\frac{1}{n}})}{\approx} \frac{1}{w} \frac{\mu(D_t)}{vol(\mathbb{R}^n/\pi(y))}$$

Need to express this in terms of volumes...

Lemma: $B \subseteq \mathbb{R}^n$ bdd, $\partial B$ $(n-1)$-Lipschitz parametrizable (later)

$\Lambda \subseteq \mathbb{R}^n$ lattice

$$\Rightarrow \#(\Lambda \cap a \cdot B) = \frac{\mu(B)}{vol(\mathbb{R}^n/\Lambda)} \cdot a^n + O(a^{n-1}), \quad a \ge 1$$



~~M~~ Apply this to

$B = D_1$ , $a = t^{\frac{1}{n}}$ $\left(\Rightarrow aB = t^{\frac{1}{n}} \cdot D_1 \underset{\uparrow}{=} D_t\right)$

not true for the $D_t$ defined by Tim)

$$\Rightarrow \#(n(\mathfrak{y}) \cap D_t) \underset{\nearrow}{\approx} \frac{2^{r_2} \cdot \mu(D_1) \cdot t}{\sqrt{|\Delta_K|} \cdot N(\mathfrak{y})}$$

up to $O(a^{n-1}) = O(t^{\frac{n-1}{n}})$

La: $\mu(D_1) = 2^{r_1} \cdot \pi^{r_2} \cdot R_K \cdot N(\mathfrak{y})$

($\Rightarrow$ Proof for $N_C(t)$ finished up to prev. La)

Prf: $$\mu(D_1) = \int_{D_1} dy_1 \cdot \ldots \cdot dy_{r_1}\, dz_1 \cdot \ldots \cdot dz_{r_2}$$

$$\underset{\substack{\uparrow \\ \text{pol. coord.}}}{=} \int_{D_1} dy_1 \cdot \ldots \cdot dy_{r_1} \cdot \varrho_1 \cdot \ldots \cdot \varrho_{r_2} \cdot d\varrho_1 \cdot \ldots \cdot d\varrho_{r_2}$$

$$d\theta_1 \cdot \ldots \cdot d\theta_{r_2} \; , \; \varrho_i = |z_i|$$

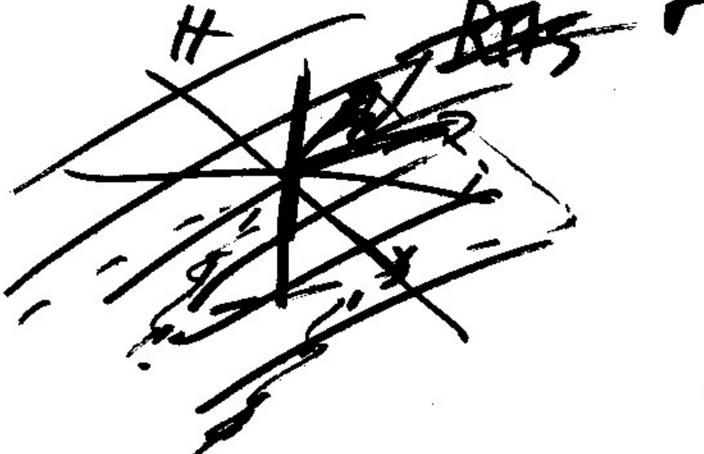
$$= 2^{r_1} \cdot (2\pi)^{r_2} \int_{D_1'} dy_1 \cdots dy_{r_1} \cdot \rho_1 \cdots \rho_{r_2} \, d\rho_1 \cdots d\rho_{r_2} =: (*)$$

$$D_1 = D_1' \times (S^1)^{r_2} \times \{\pm 1\}^{r_1}$$

$$D_1' = D_1 \cap (\mathbb{R}_{>0})^{r_1} \times (\mathbb{R}_{>0})^{r_2}$$

**Note:** $\mathrm{Log}: D_1' \xrightarrow{\sim} \{ t_0 \cdot (\overbrace{\tfrac{1}{n}, \ldots, \tfrac{1}{n}}^{r_1}, \overbrace{\tfrac{2}{n}, \ldots, \tfrac{2}{n}}^{r_2}) + t_1 \ell(u_1) + \ldots + t_s \ell(u_s) \mid t_0 \in (-\infty, \log N(y)),\ t_1, \ldots, t_s \in [0,1) \} =: D'_{\log}$

$$\mathrm{Log}(x, z) = (\log|x_i|, 2 \cdot \log|z_j|)$$




$$\Rightarrow (*) = 2^{r_1} \cdot (2\pi)^{r_2} \int_{D'_{\log}} e^{\sum_{i=1}^{r_1} x_i} \cdot e^{\frac{1}{2}\sum_{j=1}^{r_2} x_{r_1+j}} \cdot e^{\frac{1}{2}\sum_{j=1}^{r_2} x_{r_1+j}} \left(\tfrac{1}{2}\right)^{r_2} dx_1 \cdots dx_{r_1+r_2}$$

$y_i = e^{x_i}$, $i = 1, \ldots, r_1$, $\rho_j = e^{\frac{1}{2} x_j}$, $j = 1, \ldots, r_2$

(13)

$$= 2^{r_1} \cdot (2\pi)^{r_2} \int_{D'_{\log}} e^{\sum_{i=1}^{r_1+r_2} x_i} \, dx_1 \cdots dx_{r_1+r_2} =: (**)$$

Now: $(-\infty, \log N(y)) \times [0,1]^s \xrightarrow[\sim]{A} D'_{\log}$

where $A = \begin{pmatrix} 1/n & & & \\ \vdots & & & \\ 1/n & \ell(u_1) & \cdots & \ell(u_s) \\ 2/n & & & \\ \vdots & & & \\ 2/n & & & \end{pmatrix}$

Define coord. $\begin{pmatrix} v_1 \\ \vdots \\ v_{r_1+r_2} \end{pmatrix} := A^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_{r_1+r_2} \end{pmatrix}$

$$\Rightarrow e^{\sum_{i=1}^{r_1+r_2} x_i} = e^{(1,\dots,1) A \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_{r_1+r_2} \end{pmatrix}}$$

$$= e^{(1,0,\dots,0) \cdot A \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_{r_1+r_2} \end{pmatrix}} = e^{v_1}$$

&

$$(**) = 2^{r_1} \cdot (\pi)^{r_2} \int_{v_1 \in (-\infty, \log N(J)]} \int_{v_i \in [0,1]} e^{v_1} \cdot \underbrace{|\det(A)|}_{R_K = \left|\det\left(\frac{1}{r_1+r_2}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}, \ldots\right)\right|} \cdot dv_1 \cdot \ldots \cdot dv_{r_1+r_2}$$

$$= 2^{r_1} \pi^{r_2} \cdot N(J) \cdot R_K \qquad \square$$

Return to lemma:

Def: 1) $f : [0,1]^{n-1} \to \mathbb{R}^n$ Lipschitz

if $\dfrac{|f(x) - f(y)|}{|x - y|}$ uniformly bdd,

$x, y \in [0,1]^{n-1}$

2) $B \subseteq \mathbb{R}^n$ bdd, $\partial B = \overline{B} \setminus B^\circ$

$\partial B$ is $(n-1)$-Lipschitz parametrizable if $\partial B$ is covered by images of fin. many Lipschitz fcts $f: [0,1]^{n-1} \to \mathbb{R}^n$

Aim: $$\#(\Lambda \cap aB) = \frac{\mu(B)}{vol(\mathbb{R}^n/\Lambda)} \cdot a^n + O(a^{n-1})$$

Wlog $\Lambda = \mathbb{Z}^n$ (by lin. tr $L: \mathbb{R}^n \to \mathbb{R}^n$ with $L(\mathbb{Z}^n) = \Lambda$

$\Rightarrow |\det(L)| = vol(\mathbb{R}^n/\Lambda)$,

$\mu(L^{-1}(B)) = \frac{1}{|\det(L)|} \cdot \mu(B)$ )

Set $D = [0,1]^n$

Set $n_0(a) := \#\{ x \in \Lambda \mid x + D \subseteq aB \}$

$$\begin{array}{c} n_0(a) \\ \wedge| \\ \#(\Lambda \cap aB) \\ \wedge| \\ n_1(a) \end{array}$$

$n_1(a) := \#\{ x \in \Lambda \mid x + D \cap aB \neq \emptyset \}$

$|n_1(t) - n_0(t)|$

$\leq \#\{\pi \in \mathcal{L} \mid \pi + D \cap \partial B \neq \emptyset\}$

Pick ~~$\ldots$~~ s.t. $f_1, \ldots, f_m : [0,1]^{n-1} \to \mathbb{R}^n$

Lipschitz with $\bigcup \operatorname{Im} f_i = \partial B$

Pick $c_i > 0$, s.t.

$$\frac{|f_i(x) - f_i(y)|}{|x - y|} < c_i \quad \forall x, y \in [0,1]^{n-1}$$

Let $c := \sqrt{n} \max_{i=1,\ldots m} \{c_i\}$

Claim: Each $y \in \partial B$ lies within distance $c/a$ to a point in

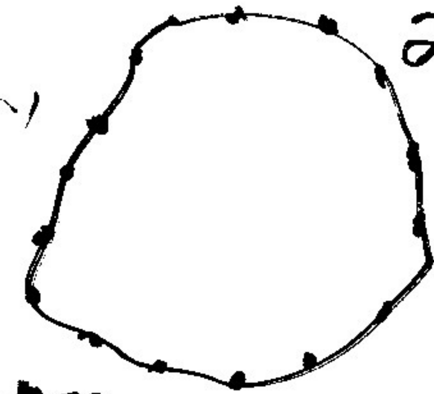
$$P := \left\{ f_i\left(\frac{r_1}{a}, \ldots, \frac{r_{n-1}}{a}\right) \,\middle|\, \begin{array}{l} 1 \leq i \leq m,\ r_i \in \mathbb{Z} \\ 0 \leq r_1, \ldots, r_{n-1} \leq a \end{array} \right\}$$

next time

$P$

finite

set



$\partial B$

$\# P \leq m \cdot (a+1)^{n-1}$

Given $n$, $a \in \mathbb{Z}$, s.t. $(a,n)=1$

$\Rightarrow \exists$ inf. primes $p$, s.t. $p \equiv a \ (n)$